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Robustness in the Pareto-solutions for the Multi-Criteria Minisum Location Problem

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ABSTRACT

In this paper, a new trend is introduced into the field of multi-criteria location problems. We combine the robustness approach using the minmax regret criterion together with Pareto-optimality. We consider the multi-criteria squared Euclidean minisum location problem which consists of simultaneously minimizing a number of weighted sum-distance functions and the set of Pareto-optimal locations as its solution concept. The Pareto-optimal solutions for the set of robust locations with respect to the original weighted sum-distance functions is completely characterized. These Pareto-optimal solutions have both the properties of stability and non-domination which are required in robust and multi-criteria programming.

KEY WORDS: multi-criteria location; robustness; minmax regret

1. INTRODUCTION

In the last few years a trend has become very important in the field of optimization: robust optimization. There are different reasons for considering robustness and possibly the most important one is that it helps to consider uncertainty. Uncertainty affects a great variety of decision processes such as cost or production processes, investment decisions, inventory management, scheduling or demand forecasting among others.

There is a wide range of criteria for handling decisions for uncertainty models. One can mention the deterministic optimization approach, the stochastic optimization and the robust approach. In the first one, the decision maker 'chooses' one instance of the input data and then solves the model for this specific choice. In the second one, some kind of information about the potential occurrences of the data in the future is estimated and the model will attempt to generate a solution that maximizes (or minimizes) an expected effectiveness criterion. The main drawback of these two approaches is that the input data of both lead to a whole range of feasible solutions, so that either the most probable (likely) or the expected data scenario does not cover all of them.

In the robust approach the aim is to produce a solution that behaves acceptably well under any likely input data. Among the different criteria that can be used to manage robustness we will use the minmax regret. It consists of minimizing the 'regret' or difference between the objective value of a feasible solution and the optimal solution that would have been chosen if the decision-maker would have known the actual input data (see Kouvelis and Yu, 1997, for further details on this kind of analysis).

In this paper, we consider the single facility location problem under the viewpoint of uncertainty. In this framework, the uncertainty is driven by the different location scenarios that may occur. We will consider that it is agreed on that the quality of a location will be given by weighted-sum objective functions. The uncertainty is given by assigning each existing facility a set of weights representing a different scenario. In addition, we will assume that different decision-makers, each

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having different scenarios to compare, interact. Since facility location decisions usually involve a serious amount of money it is reasonable to assume that not a single person but a group of (equally ranked) persons has to decide. In this situation, the proposed solution has to be a compromise between the involved decision-makers. To fulfil this requisite we propose the Pareto solutions with respect to the robust criteria controlled by the decision-makers. The main goal of this paper is to give a complete geometrical description of the whole set of Pareto-optimal solutions with respect to several minmax regret criteria. These solutions are: (1) robust because they result from the regret criterion and (2) Pareto, therefore they are not dominated componentwise.

This model also has another interpretation. It can be seen as an intermediate situation between multifacility and single facility location, which consists of locating k different servers based at a unique centre and only once. Nevertheless, the determination of such a point cannot be described by the classical criteria (sum, maximum, etc.) through aggregation because each server has its own interest and thus its own scenario to be considered. Each server wants its objective value to be as close as possible to its optimal value. Hence, this model leads to a problem where we look for a location minimizing the maximum deviation of each objective regarding the optimal objective value for each one of the servers, i.e. minmax regret with respect to the different scenarios:

$$\min_{x} \max_{w \in \{w^1, \ldots, w^k\}} [f_w(x) - f_w(x(w))]$$

where w is the set of parameters which specifies a certain scenario, $\{w^1, \ldots, w^k\}$ are the different scenarios to be considered, f_w is the objective function under the scenario w and $x(w)$ is an optimal solution (minimum) of the problem with objective function f_w.

Furthermore, it may occur that each server behaves differently in several time periods. Thus, we also have different scenarios to consider in each time period. Since only one location is allowed for all the periods each server may consider its own problem as a multi-criteria problem. The server must find those solutions not dominated in the objective values with respect to the time horizon since nobody wants a locational decision which can be improved in all time periods under consideration simultaneously.

Combining both features we obtain again the multi-criteria minmax regret. This methodology could be naturally applied to the real-world situation described in the report on 'Stationing of Rescue helicopters in Southern Tirol' (Ehrgott, 1998). There, the case of three helicopters to be based at a common location is considered and different strategies are used.

In order to obtain a description of the solution set we first solve the bicriteria problem giving a polynomial time algorithm to obtain the bicriteria Pareto chain. Then, we reduce the Q-criterion problem to the determination of all the Pareto-optimal solutions for any subset of 3-criterion. Finally, we show how to characterize these sets using only bicriteria Pareto-solutions chains.

The paper is organized as follows. In Section 2 we introduce the single objective minmax regret location problem and state an equivalent easier formulation. The geometrical structure of the optimal solution set of the single objective minmax regret location problem is analysed in Section 3. In Section 4 we characterize the set of Pareto solutions of the bicriteria minmax regret location problem and we formulate an algorithm to compute them. Section 5 presents a complete description of the set of Pareto solutions in the Q-criteria case using convex analysis and the results of the previous section. The paper ends with some conclusions and an outlook to further research.

2. THE MODEL

Let A be a denumerable set of existing facilities and W a finite set of weight vectors $w \in \mathbb{R}^{|A|}$. Each $w \in W$ satisfies $\sum_{a \in A} w_a = 1$ and $w_a \geqslant 0, \forall a \in A$. In other words, $w \in W$ represents a location scenario for a decision maker (DM) while w_a is the importance given to the existing facility $a \in A$ in the scenario w. We assume that distances are measured by the squared Euclidean norm. Several examples of such a quadratic formulation are the problems of locating hospitals, fire stations, police stations, and other emergency service agencies. In this cases, the damage incurred increases more than proportionally with the waited time for intervention, hence with the distance travelled (see White, 1971; Ohsawa 1999). In addition, since the quadratic models can be considered as an approximation to the Euclidean distance cases, such a quadratic formulation is justified (see

McHose, 1961). Therefore, our minmax regret problem for a single DM is

$$\min_{x\in\mathbb{R}^2} \max_{w\in W} \left[\sum_{a\in A} w_a \|x-a\|_2^2 - \sum_{a\in A} w_a \|x(w)-a\|_2^2 \right] \quad (1)$$

where $x(w)$ is the optimal solution of problem (2):

$$\min_x \sum_{a\in A} w_a \|x-a\|_2^2 \quad (2)$$

It is well-known that $x(w) = \sum_{a\in A} w_a a / \sum_{b\in A} w_b$. Besides, since we have taken normalized weights $x(w) = \sum_{a\in A} w_a a$. This fact leads us to reformulate problem (1) as

$$\min_{x\in\mathbb{R}^2} \max_{w\in W} \left[\sum_{a\in A} w_a \|x-a\|_2^2 - \sum_{a\in A} w_a \| \sum_{b\in A} w_b b - a\|_2^2 \right] \quad (3)$$

We can simplify this formulation even more by using properties of the scalar product.

Lemma 2.1
Problem (3) is equivalent to

$$\min_{x\in\mathbb{R}^2} \max_{w\in W} F_w(x) := \|x - x(w)\|_2^2 \quad (4)$$

Proof
The objective function of problem (3) can be rewritten using the scalar product $\langle \cdot, \cdot \rangle$ as

$$\begin{aligned}
&\sum_{a\in A} w_a \|x-a\|_2^2 - \sum_{a\in A} w_a \| \sum_{b\in A} w_b b - a\|_2^2 \\
&\quad = \sum_{a\in A} w_a [\langle x-a, x-a\rangle - \langle x(w)-a, x(w)-a\rangle] \\
&\quad = \sum_{a\in A} w_a [\langle x, x\rangle - 2\langle x, a\rangle \\
&\qquad - \langle x(w), x(w)\rangle + 2\langle x(w), a\rangle] \\
&\quad = \langle x, x\rangle - 2\langle x, x(w)\rangle + \langle x(w), x(w)\rangle \\
&\quad = \|x - x(w)\|_2^2 \qquad (5)
\end{aligned}$$

Therefore, both problems are equivalent. □

In the following we denote by $\mathcal{X}^*(W)$ the optimal solution of problem (4).

3. THE SINGLE OBJECTIVE REGRET LOCATION PROBLEM

We begin this section by studying some properties of the objective function of problem (4):

$$R_W(x) := \max_{w\in W} F_w(x) \quad (6)$$

Proposition 3.1
The function $R_W(x)$ is a strictly convex function.

The proof is straightforward.

The solution of problem (4) can be found by solving the equivalent convex problem with linear objective

$$\begin{aligned}
\min \quad & z \\
\text{s.t.} \quad & \|x - x(w)\|_2^2 - z \leqslant 0 \quad \forall w \in W \qquad (7)\\
& z \geqslant 0, \quad x \in \mathbb{R}^2
\end{aligned}$$

Since (7) is a convex problem, we can apply the Kuhn–Tucker conditions and an optimal solution can be derived by solving the system:

$$\begin{bmatrix} 1 - \sum_{w\in W} \lambda_w \\ \sum_{w\in W} \lambda_w (x - x(w)) \\ \lambda_w (\|x - x(w)\|_2^2 - z) \quad \forall w \in W \end{bmatrix} = 0$$

$$\Leftrightarrow \mathcal{X}^*(W) = \frac{\sum_{w\in J(\mathcal{X}^*(W))} \lambda_w x(w)}{\sum_{w\in J(\mathcal{X}^*(W))} \lambda_w} \quad (8)$$

for some choice of $\{\lambda_w\}_{w\in W}$ and $J(\mathcal{X}^*(W)) = \{w \in W : \|\mathcal{X}^*(W) - x(w)\|_2^2 = \max_{\mu\in W} \|\mathcal{X}^*(W) - x(\mu)\|_2^2\}$.

It is worth noting that problem (4) is a usual minmax location problem with respect to the new set of existing locations $\{x(w) : w \in W\}$. Therefore, there are also specific methods in the literature to solve this problem such as the well-known Elzinga–Hearn algorithm (Elzinga and Hearn, 1972).

The max operator induces a cell subdivision in the decision space of the problem. For each $\mu \in W$ consider the set:

$$C_\mu = \{x \in \mathbb{R}^2 : F_\mu(x) \geqslant F_w(x) \quad \forall w \in W\} \quad (9)$$

The sets C_μ are the farthest-point Voronoi diagrams with respect to the functions F_μ. See Okabe *et al.* (1992) for algorithms to compute farthest point Voronoi diagrams. Therefore, these sets are important because within C_μ the objective function $R_W(x)$ of problem (4) coincides with $F_\mu(x)$. Hence, provided that the geometry of these sets is easy, problem (4) reduces to solving a finite number of classical covering circle problems, one on each of these regions. The following result proves that these sets are polyhedra, thus, easy to handle.

Proposition 3.2
C_μ is a polyhedron for any $\mu \in W$.

Proof
The set C_μ is described by the following family of inequalities $F_\mu(x) - F_w(x) \geqslant 0 \quad \forall w \in W$. Now, we have that

$$\begin{aligned} F_\mu(x) - F_w(x) &= \langle x - x(\mu), x - x(\mu)\rangle \\ &\quad - \langle x - x(w), x - x(w)\rangle \\ &= 2\langle x, x(w) - x(\mu)\rangle + \langle x(\mu), x(\mu)\rangle \\ &\quad - \langle x(w), x(w)\rangle \end{aligned}$$

which is a linear function in x. Therefore, C_μ is a region bounded by linear inequalities. Hence, it is a polyhedron. □

The following result characterizes the optimal solution of problem (4) within the region C_μ. Let us denote by $\mathscr{X}^*(W,\mu)$ the optimal solution within C_μ and let $J(\mu) := \{w \in W : F_\mu(\mathscr{X}^*\ (W,\mu)) - F_w(\mathscr{X}^*(W,\mu)) = 0\}$.

Lemma 3.1
The explicit form of the optimal solution of problem (4) within C_μ is given by the following statements:

1. If $\mathscr{X}^*(W,\mu)$ belongs to the interior of C_μ then $\mathscr{X}^*(W,\mu) = x(\mu)$.
2. If $\mathscr{X}^*(W,\mu)$ does not belong to the interior of C_μ then
$$\mathscr{X}^*(W,\mu) = x(\mu) + \sum_{w \in J(\mu)} \lambda_w (x(w) - x(\mu))$$
for some $\lambda_w \geqslant 0$

Proof
Within C_μ, problem (4) can be described as

$$\min_{x=(x_1,x_2)\in\mathbb{R}^2} \quad \langle x - x(\mu), x - x(\mu)\rangle \tag{10}$$

$$\text{s.t.} \quad 2\langle x, x(\mu) - x(w)\rangle \leqslant \|x(\mu)\|_2^2 - \|x(w)\|_2^2 \quad \forall w \in W \tag{11}$$

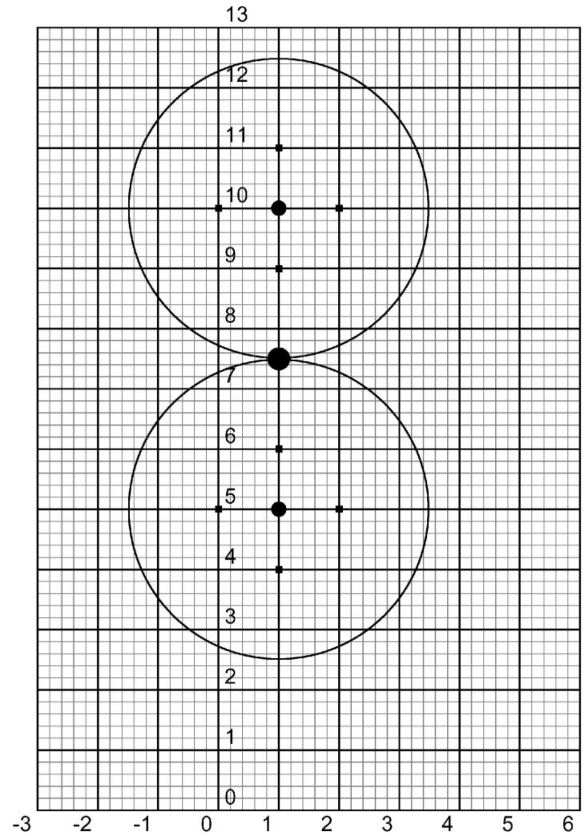

Figure 1. Illustration of Example 3.1. The eight small squares represent the existing facilities. The two middle sized points indicate the optimal solutions for the weight sets w^1 and w^2, respectively. The largest point represents the optimal solution $\mathscr{X}^*(W)$ for the two weight scenarios and the radius of the two circles illustrate the corresponding optimal objective value.

Then, an optimal solution $\mathscr{X}^*(W,\mu)$ within this region can be obtained using the Kuhn–Tucker conditions:

$$(x - x(\mu)) + \textstyle\sum_{w\in W} \lambda_w (x(\mu) - x(w)) = 0$$
$$\lambda_w(\langle x, x(\mu) - x(w)\rangle - \|x(\mu)\|_2^2 + \|x(w)\|_2^2) = 0 \quad \forall w \in W$$
$$\lambda_w \geqslant 0 \quad \forall w \in W$$

These conditions lead us to the explicit form of the optimal solution.

1. If $\mathscr{X}^*(W,\mu)$ belongs to the interior of C_μ then only $\lambda_\mu \neq 0$. Therefore, $\mathscr{X}^*(W,\mu) = x(\mu)$.

2. If $\mathcal{X}^*(W, \mu)$ does not belong to the interior of C_μ, let us denote by $J(\mu) := \{w \in W : F_\mu(\mathcal{X}^*(W, \mu)) - F_w(\mathcal{X}^*(W, \mu)) = 0\}$. In this case, one has

$$\mathcal{X}^*(W, \mu) = x(\mu) + \sum_{w \in J(\mu)} \lambda_w (x(w) - x(\mu))$$

for some $\lambda_w \geqslant 0$. □

Example 3.1
We are given 8 existing facilities $a_1 = (1, 11)$, $a_2 = (1, 9)$, $a_3 = (0, 10)$, $a_4 = (2, 10)$, $a_5 = (1, 6)$, $a_6 = (1, 4)$, $a_7 = (0, 5)$ and $a_8 = (2, 5)$.

We have two sets of weights $w^1 = (1, 1, 1, 1, 0, 0, 0, 0)$ and $w^2 = (0, 0, 0, 0, 1, 1, 1, 1)$, $W = \{w^1, w^2\}$. For the sake of readability we do not normalize the weight vectors. First, we compute the optimal solution for each weight set by

$$\frac{\sum a_i w_i^1}{\sum w_i^1} = \frac{(4, 40)}{4} = (1, 10) = x(w^1)$$

and

$$\frac{\sum a_i w_i^2}{\sum w_i^2} = \frac{(4, 20)}{4} = (1, 5) = x(w^2)$$

Now we find $\mathcal{X}^*(W)$ by computing the midpoint of the segment $[(1, 10), (1, 5)]$ which is $(1, 7.5)$. The results are shown in Figure 1.

4. THE BICRITERIA REGRET LOCATION PROBLEM

Consider two decision makers each one of them having a set of different scenarios and wishing to make a decision looking for a compromise among themselves. Each DM has a set of weights W^k, $k = 1, 2$, and the bicriteria problem is

$$\min_{x \in \mathbb{R}^2} \left[\max_{w^1 \in W^1} \|x - x(w^1)\|_2^2, \max_{w^2 \in W^2} \|x - x(w^2)\|_2^2 \right] \quad (12)$$

Let us denote by $\mathcal{X}^*_{\text{par}}(W^1, W^2)$ the Pareto-optimal solution set of problem (12), by $\mathcal{BI}(W^k)$ the set of orthogonal bisectors of the points $x(w^{ki})$, $x(w^{kj})$ for all $w^{ki} \neq w^{kj} \in W^k$ with $k = 1, 2$ and by $\mathcal{SEG}(W^1, W^2) := \bigcup_{w^1 \in W^1; w^2 \in W^2} [x(w^1), x(w^2)]$, i.e. the set of line segments joining the points $x(w^1)$ with $x(w^2)$ for any $w^1 \in W^1$ and $w^2 \in W^2$.

Let $C(W^1, W^2)$ be the superposition (intersection) of the two cell subdivisions $C(W^1)$ and $C(W^2)$ which were defined in (9). This is to say,

$$C(W^1, W^2) = \{C_{w^1, w^2} := C_{w^1} \cap C_{w^2} : w^1 \in W^1,\ w^2 \in W^2\}$$

Within a set C_{w^1, w^2}, problem (12) reduces to

$$\min_{x \in C_{w^1, w^2}} \{\|x - x(w^1)\|_2^2, \|x - x(w^2)\|_2^2\}$$

Since the squared Euclidean distance is an increasing one-to-one transformation of the Euclidean distance, the Pareto-optimal solution set of our problem coincides with the Pareto-optimal solution set of the Euclidean point-objective location problem (with only two points). For this problem it is known (see Carrizosa *et al.*, 1993) that its set of Pareto-optimal solutions consists of the orthogonal projection of the convex hull (line segment for the case of only two points) of the existing facilities onto the constraint set. Let us denote by $\mathcal{X}^*_{\text{par}}(w^1, w^2; C_{w^1, w^2})$ the set of Pareto-optimal solutions of problem (12) in C_{w^1, w^2}. Therefore, using the mentioned equivalence we conclude that

$$\mathcal{X}^*_{\text{par}}(w^1, w^2; C_{w^1, w^2}) = \text{proj}_{C_{w^1, w^2}}([x(w^1), x(w^2)]) \quad (13)$$

where $\text{proj}_X(a)$ is the orthogonal projection of a onto X. It is worth noting that some parts of the projection may coincide with the line segment when this intersects the considered region. Therefore, all the Pareto-optimal solutions of problem (12) are on the boundary of C_{w^1, w^2} or on $[x(w^1), x(w^2)] \cap C_{w^1, w^2}$ as shown in the next lemma.

Lemma 4.1

$$\mathcal{X}^*_{\text{par}}(W^1, W^2) \subset \mathcal{BI}(W^1) \cup \mathcal{BI}(W^2) \cup \mathcal{SEG}(W^1, W^2).$$

Proof
Since $C(W^1, W^2)$ is a subdivision of $\mathbb{R}^2$ one has that

$$\mathcal{X}^*_{\text{par}}(W^1, W^2) \subset \bigcup_{w^1 \in W^1,\ w^2 \in W^2} \mathcal{X}^*_{\text{par}}(w^1, w^2; C_{w^1, w^2}) \quad (14)$$

Then, just note that by (13):

$$\mathcal{X}^*_{\text{par}}(w^1, w^2; C_{w^1, w^2}) \subset \mathcal{BI}(W^1) \cup \mathcal{BI}(W^2) \cup \mathcal{SEG}(W^1, W^2) \quad (15)$$

Combining (14) and (15) the result follows. □

As a consequence of this result we get,

Lemma 4.2
$\mathcal{X}^*_{\text{par}}(W^1, W^2)$ is a connected polygonal chain on $\mathcal{BI}(W^1) \cup \mathcal{BI}(W^2) \cup \mathcal{SEG}(W^1, W^2)$ with end-points at $\mathcal{X}^*(W^1)$ and $\mathcal{X}^*(W^2)$.

The proof is a straightforward consequence of Lemma 4.1 and the results on connectivity of Pareto-solution sets for convex multiobjective programming (see Warburton, 1983).

Applying this result we can develop an algorithm for solving problem (12). In order to do that, we will need to check whether or not a particular point x is Pareto-optimal. The function *condition*(x) which takes the values *true* or *false* performs this operation. This function is defined in the following lemma. Let us denote by $\text{int}(A)$ and $\partial(A)$ the interior and the boundary of the set A, respectively.

Lemma 4.3

1. $x \in \text{int}(C_{w^1,w^2})$ for some $w^1 \in W^1$ and $w^2 \in W^2$.

$$\text{condition}(x) = \begin{cases} \text{true} & \text{if } \dfrac{x - x(w^1)}{\|x - x(w^1)\|_2} = -\dfrac{x - x(w^2)}{\|x - x(w^2)\|_2} \\ \text{false} & \text{otherwise} \end{cases}$$

2. $x \in \partial(C_{w^1,w^2})$ for some $w^1 \in W^1$ and $w^2 \in W^2$. Let $J^k(x) := \{\lambda \in W^k : F_{w^k}(x) = \|x - x(\lambda)\|_2^2\}$ for $k = 1, 2$

$$\text{condition}(x) = \begin{cases} \text{true} & \text{if } 0 \in \text{conv}\Big\{\bigcup_{\lambda \in J^1(x)} (x - x(\lambda)) \cup \bigcup_{\mu \in J^2(x)} (x - x(\mu))\Big\} \\ \text{false} & \text{otherwise} \end{cases}$$

Proof

Case 1. According to (13), $x \in \text{int}(C_{w^1,w^2})$ is a Pareto-solution if it belongs to the line segment $[x(w^1), x(w^2)]$. This also means that x is an unconstrained Pareto solution and therefore, the gradients of the two-objective functions must be opposite. This fact proves the expression of condition(x) in Case 1.

Case 2. First note that since the objective functions are strictly convex the Pareto-solutions coincide with weak Pareto-solutions (see e.g. White, 1982). Therefore, $x \in \partial(C_{w^1,w^2})$ is a Pareto-solution if and only if it fulfils the Karush–Kuhn–Tucker weak Pareto-optimality condition for non-differentiable convex functions: "zero belongs to convex hull of the union of the subdifferential sets of the two objective functions at x" (see e.g. Miettinen, 1999). On the boundary, the objective functions are the pointwise maximum of squared Euclidean distances. The subdifferential set of the maximum is the convex hull of the subdifferential sets of those functions achieving the maximum at the considered point x. This is exactly the expression of the function in Case 2. □

Algorithm 4.1

<u>*Input:*</u>
1. *Demand points* $A \subset \mathbb{R}^2$.
2. *Weight sets* $W^1 = (w_a^1)_{a \in A}$ *and* $W^2 = (w_a^2)_{a \in A}$.

<u>*Output :*</u>
1. $\mathcal{X}^*_{\text{par}}(W^1, W^2)$.

<u>*Steps:*</u>
1. *Computation of the planar graph generated by* $C(W^1, W^2) \cup \mathcal{SEG}(W^1, W^2)$.
2. *Compute the optimal solutions of the single criterion problems:* $\mathcal{X}^*(W^1)$ *and* $\mathcal{X}^*(W^2)$.
3. *IF* $\mathcal{X}^*(W^1) = \mathcal{X}^*(W^2)$
4.   *THEN* (★ *trivial case* ★)
5. $\mathcal{X}^*_{\text{par}}(W^1, W^2) := \mathcal{X}^*(W^1)$
6.   *ELSE* (★ *non trivial case* ★)
7. $\mathcal{X}^*_{\text{par}}(W^1, W^2) := \mathcal{X}^*(W^1) \cup \mathcal{X}^*(W^2)$
8.   *Choose* $x := \mathcal{X}^*(W^1)$.
9.   *WHILE* $x \neq \mathcal{X}^*(W^2)$ *DO*
10.   *BEGIN*
11.   *REPEAT*
12.   *Choose* $y \in Adj(x)$ (★*Adj*(x) *is the set of adjacent vertices to* x★)
13.   *UNTIL* condition(y)
14. $\mathcal{X}^*_{\text{par}}(W^1, W^2) := \mathcal{X}^*_{\text{par}}(W^1, W^2) \cup \overline{xy}$
15. $x := y$
16.   *END*

If we analyse the complexity, we first recognize that the optimal solutions for each $w \in W^i$,

$i = 1, 2$, can be computed with (8) in linear time with respect to $|w|$. Also, the optimal solutions $\mathcal{X}^*(W^i)$, $i = 1, 2$, for a single DM can be computed in linear time with respect to $|W^i|$ (see Megiddo, 1982). For the computation of $\mathcal{X}^*_{\text{par}}(W^1, W^2)$ we need to determine the planar graph induced by $C(W^1, W^2)$ and $\mathcal{SEG}(W^1, W^2)$. Using a scan-line-principle (Bentley and Ottmann, 1979) proved that the computation of a planar graph induced by n line segments in the plane, can be obtained in $O((n + s)\log n)$ time, where s is the number of intersection points of the line segments. In this case, it means that this process can be done in $O(K^2 \log K)$, where $K = \max(|W^1|, |W^2|)$. The evaluation of *condition* can be done in linear time with respect to K (see Frenk *et al.*, 1996). Since we have not more than $O(K^2)$ vertices in our planar graph the total complexity is $O(K^3 \log K)$.

Example 4.1

We use the data of Example 3.1 and add 8 additional existing facilities $a_9 = (8, 9)$, $a_{10} = (8, 7)$, $a_{11} = (7, 8)$, $a_{12} = (9, 8)$, $a_{13} = (10, 2)$, $a_{14} = (10, 0)$, $a_{15} = (9, 1)$ and $a_{16} = (11, 1)$.

Now we have two decision makers, each of them having two sets of weights:

$W^1 = \{w^{11}, w^{12}\}$ with

- $w^{11} = (1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ and
- $w^{12} = (0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0)$

$W^2 = \{w^{21}, w^{22}\}$ with

- $w^{21} = (0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0)$ and
- $w^{22} = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1)$.

Analogous to Example 3.1 we get the optimal single criterion solutions $x(w^{11}) = (1, 10)$, $x(w^{12}) = (1, 5)$, $\mathcal{X}^*(W^1) = (1, 7.5)$, $x(w^{21}) = (8, 8)$, $x(w^{22}) = (10, 1)$ and $\mathcal{X}^*(W^2) = (9, 4.5)$. Next, we compute the set of Pareto solutions $\mathcal{X}^*_{\text{par}}(W^1, W^2)$ starting at $\mathcal{X}^*(W^1)$ (see also Figure 2). According to Algorithm 4.1 we test an adjacent vertex to $\mathcal{X}^*(W^1)$ in the planar graph induced by $C(W^1, W^2)$. The only choice is the point $P_1 = (3.52, 7.5)$. We have Case 2 of Lemma 4.3 and therefore we have to check whether

$$0 \in \operatorname{conv}\left\{ \bigcup_{\lambda \in J^1(x)} (P_1 - x(\lambda)) \cup \bigcup_{\mu \in J^2(x)} (P_1 - x(\mu)) \right\}$$

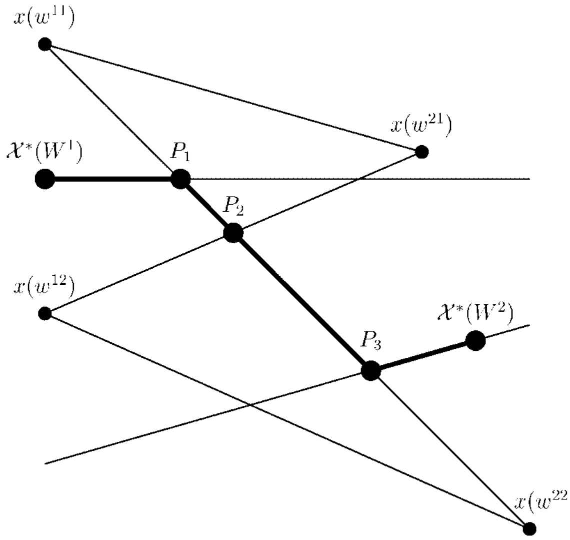

Figure 2. Illustration of Example 4.2. The bold part consitutes the set of Pareto solutions.

We see that P_1 is on the bisector between $x(w^{11})$ and $x(w^{12})$. Therefore, $J^1(P_1) = \{w^{11}, w^{12}\}$ and $J^2(P_1) = \{w^{22}\}$. Since P_1 is on the line segment connecting $x(w^{11})$ and $x(w^{22})$ we know that we have 0 already in the convex hull of $P_1 - x(w^{11})$ and $P_1 - x(w^{22})$. Therefore, P_1 belongs to $\mathcal{X}^*_{\text{par}}(W^1, W^2)$. If we would continue on the bisector between $x(w^{11})$ and $x(w^{12})$, we would still have the same sets J^1 and J^2 but we would need $x(w^{21})$ for the convex hull construction. This means that there is no Pareto solution in this direction and we have to continue with point $P_2 = (4.5, 6.5)$. Now we are in the interior of $C_{w^{11}, w^{22}}$ and we have to test Case 1 of Lemma 4.3 which is fulfilled since P_2 is also on the line segment joining $x(w^{11})$ and $x(w^{22})$. Therefore, P_2 is in the set of Pareto solutions and the unique adjacent vertex is P_3 from where we have as an adjacent vertex already $\mathcal{X}^*(W^2)$ and we are done.

5. THE MULTI-CRITERIA REGRET LOCATION PROBLEM

In this section, we turn to the Q-criteria case and we will develop an efficient algorithm for computing $\mathcal{X}^*_{\text{par}}(W^1, \ldots, W^Q)$ using the results of the bicriteria case. In order to obtain a geometrical characterization of a Pareto solution we use convex analysis.

For a multiobjective problem let $\mathscr{X}^*_{w-\text{par}}$ denote the set of weak-Pareto solutions. Using the level sets and level curves (Hamacher and Nickel, 1996) obtained that a point $x \in \mathbb{R}^2$ is a weak Pareto solution if and only if the following statement holds:

$$\bigcap_{q=1}^{Q} L_<(R_{W^q}, R_{W^q}(x)) = \emptyset$$

Moreover, if the objective functions are strictly convex, White (1982) proved that $\mathscr{X}^*_{\text{par}} = \mathscr{X}^*_{w-\text{par}}$.

First, we start giving the general result for the Q-criteria case which reduces the problem to the computation of the 3-criterion solution sets. This result motivates the technical details needed to characterize the three objectives case.

Theorem 5.1

$$\mathscr{X}^*_{\text{par}}(W^1, W^2, \ldots, W^Q) = \bigcup_{i,j,k} \mathscr{X}^*_{w-\text{par}}(W^i, W^j, W^k)$$

Proof

Since the objective functions R_{W^i} are strictly convex, it follows that $\mathscr{X}^*_{\text{par}}(W^1, \ldots, W^Q) = \mathscr{X}^*_{w-\text{par}}(W^1, \ldots, W^Q)$. Then, $x \in \mathscr{X}^*_{w-\text{par}}(W^1, \ldots, W^Q)$ iff $\bigcap_{1 \leqslant i \leqslant Q} L_<(R_{W^i}, R_{W^i}(x)) = \emptyset$. This intersection is empty if and only if there exist $i, j, k \in Q$ such that $L_<(R_{W^i}, R_{W^i}(x)) \cap L_<(R_{W^j}, R_{W^j}(x)) \cap L_<(R_{W^k}, R_{W^k}(x)) = \emptyset$ (see Helly's Theorem; Rockafellar (1970)) and this is equivalent to $x \in \mathscr{X}^*_{w-\text{par}}(W^i, W^j, W^k)$. Since in any case

$$\bigcup_{\substack{i,j,k \\ i \neq j \neq k}} \mathscr{X}^*_{w-\text{par}}(W^i, W^j, W^k) \subset \mathscr{X}^*_{w-\text{par}}(W^1, W^2, \ldots, W^Q)$$

the proof is complete. □

For more results on the reduction of criteria, in general, multi-criteria problems the reader is referred to Ehrgott and Nickel (2000) and references therein.

Now we proceed with the 3-criterion case. In order to do that some notation is necessary. For our function $R_W(x)$ the level and strict level sets for a value $z \in \mathbb{R}$ are given by

$$L_\leqslant(R_W, z) := \{x \in \mathbb{R}^n \ : \ R_W(x) \leqslant z\}$$

and

$$L_<(R_W, z) := \{x \in \mathbb{R}^n \ : \ R_W(x) < z\}$$

In the same way, we define the complement of the strict level set as

$$L_\geqslant(R_W, z) := \{x \in \mathbb{R}^n \ : \ R_W(x) \geqslant z\}$$

and the level curve for a value $z \in \mathbb{R}$ is given by

$$L_=(R_W, z) := \{x \in \mathbb{R}^2 \ : \ R_W(x) = z\}$$

The tangent cone $T_B(x)$ to the convex set B at point x is

$$T_B(x) := \overline{\text{cone}}(B - x)$$

where for any set S, $\bar{S}$ stands for the topological closure of S.

Let us denote

$$I_{ij}^{\leqslant}(x) := L_\leqslant(R_{W^i}, R_{W^i}(x)) \cap L_\leqslant(R_{W^j}, R_{W^j}(x))$$

$$I_{ij}^{<}(x) := L_<(R_{W^i}, R_{W^i}(x)) \cap L_<(R_{W^j}, R_{W^j}(x))$$

$$i \neq j, \quad i,j = 1,2,3$$

Recently, Rodríguez-Chía (1998) proved geometrical characterizations of Pareto-optimal solutions for general location problems. The following results are consequences of this work. We will obtain a geometrical description of the 3-criterion weak-Pareto solution set. To this end, several technical lemmas are needed. In the following, the relative interior and the relative boundary of a convex set is denoted ri and rbd, respectively.

Lemma 5.1

Whenever the statements

(a) $\bigcap_{i=1}^3 L_\leqslant(R_{W^i}, R_{W^i}(x)) = \{x\}$,
(b) $I_{ij}^<(x) \neq \emptyset \quad \forall i \neq j \in \{1,2,3\}$

hold, then

(i) $x + \bigcap_{i=1}^3 T_{L_\leqslant(R_{W^i}, R_{W^i}(x))}(x) = \{x\}$.
(ii) $\mathscr{X}^*_{w-\text{par}}(W^i, W^j) \cap (x - (T_{L_\leqslant(R_{W^i}, R_{W^i}(x))}(x) \cap T_{L_\leqslant(R_{W^j}, R_{W^j}(x))}(x))) = \emptyset, \ \forall i \neq j \in \{1,2,3\}$

Proof

The first assertion is equivalent to prove that $\bigcap_{i=1}^3 T_{L_\leqslant(R_{W^i}, R_{W^i}(x))}(x) = \{0\}$. We prove this fact by contradiction. Assume that there exists $y \neq 0$ such that $y \in \bigcap_{i=1}^3 T_{L_\leqslant(R_{W^i}, R_{W^i}(x))}(x)$, then four cases may occur:

1. $y \in \text{ri}(T_{L_\leqslant(R_{W^i}, R_{W^i}(x))}(x)), \quad i = 1,2,3$ (see Figure 3).

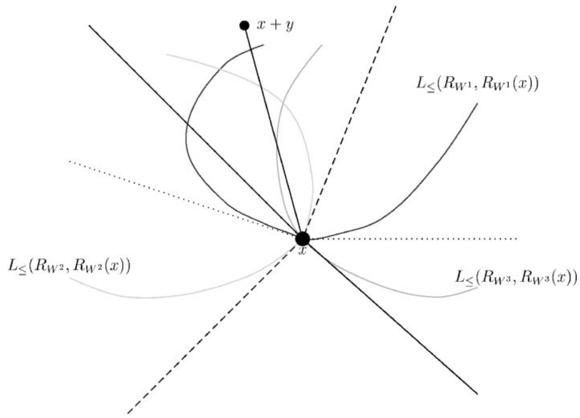

Figure 3. Case $\bigcap_{i=1}^{3} \mathrm{ri}(T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x)) \neq \emptyset$.

Since, $y \in \bigcap_{i=1}^{3} \mathrm{ri}(T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x))$, there exists $\lambda_i > 0$ such that $x + \lambda_i y \in L_\leqslant(R_{W^i}, R_{W^i}(x))$ for $i = 1,2,3$. We define $\lambda := \min\{\lambda_1, \lambda_2, \lambda_3\} > 0$. Using $x \in \bigcap_{i=1}^{3} L_\leqslant(R_{W^i}, R_{W^i}(x))$ and the convexity of $\bigcap_{i=1}^{3} L_\leqslant(R_{W^i}, R_{W^i}(x))$ we have that $x + \lambda y \in \bigcap_{i=1}^{3} L_\leqslant(R_{W^i}, R_{W^i}(x))$, and this contradicts (a).

2. $y \in \mathrm{ri}(T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x))$, $i = 1,2$ and $y \notin \mathrm{ri}(T_{L_\leqslant(R_{W^3},R_{W^3}(x))}(x))$.
Then, one of the facets of $T_{L_\leqslant(R_{W^3},R_{W^3}(x))}(x)$ belongs to $\bigcap_{i=1}^{2} \mathrm{ri}(T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x))$. Hence, we have that $\bigcap_{i=1}^{3} \mathrm{ri}(T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x)) \neq \emptyset$ and we are in Case 1.
3. $y \in \mathrm{ri}(T_{L_\leqslant(R_{W^1},R_{W^1}(x))}(x))$ and $y \notin \mathrm{ri}(T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x))$, $i = 2,3$. Then, one of the facets of $\bigcap_{i=2}^{3} T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x)$ belongs to $\mathrm{ri}(T_{L_\leqslant(R_{W^1},R_{W^1}(x))}(x))$.

Moreover, $I_{23}^{<}(x) \neq \emptyset$ then

$$\mathrm{ri}\left(\bigcap_{i=2}^{3} T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x)\right) = \bigcap_{i=2}^{3} \mathrm{ri}\left(T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x)\right) \neq \emptyset.$$

This implies that

$$\bigcap_{i=2}^{3} \mathrm{ri}\left(T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x)\right) \cap \mathrm{ri}\left(T_{L_\leqslant(R_{W^3},R_{W^3}(x))}(x)\right) \neq \emptyset$$

and we are again in Case 1.

4. $y \notin \mathrm{ri}(T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x)), i = 1,2,3$. We have that $y \in \bigcap_{i=1}^{3} T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x)$ then $y \in \mathrm{rbd}(T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x))$, $i = 1,2,3$. Hence, there exists a common facet for the three cones. Since $T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x)$ and $T_{L_\leqslant(R_{W^j},R_{W^j}(x))}(x)$ are convex and
$$\mathrm{ri}\left(T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x)\right) \bigcap \mathrm{ri}\left(T_{L_\leqslant(R_{W^j},R_{W^j}(x))}(x)\right) \neq \emptyset$$
for all $i,j \in \{1,2,3\}$, the cones $T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x)$ and $T_{L_\leqslant(R_{W^j},R_{W^j}(x))}(x)$ lie in the same halfspace generated by the common facet of the three cones. Therefore, $\cap_{i=1}^{3}\mathrm{ri}(T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x))$ is not empty and we are again in Case 1.

Now, we prove the second assertion. Let $y \in T_{I_{ij}^{\leqslant}(x)}(x)$, then $x - y \in L_\geqslant(R_{W^i}, R_{W^i}(x)) \cap L_\geqslant(R_{W^j}, R_{W^j}(x))$ because $x - T_{I_{ij}^{\leqslant}(x)}(x) \subseteq L_\geqslant(R_{W^i}, R_{W^i}(x)) \cap L_\geqslant(R_{W^j}, R_{W^j}(x))$. Thus, we have that $R_{W^k}(x) \leqslant R_{W^k}(x - y)$, $k = i,j$. On the other hand, using (b) we obtain that $x \notin \mathscr{X}^*_{w-\mathrm{par}}(W^i, W^j)$. Hence $x - y \notin \mathscr{X}^*_{w-\mathrm{par}}(W^i, W^j)$ and therefore $\mathscr{X}^*_{w-\mathrm{par}}(W^i, W^j) \cap (x - T_{I_{ij}^{\leqslant}(x)}(x)) = \emptyset$.

Since $\emptyset \neq I_{ij}^{<}(x) = L_<(R_{W^i}, R_{W^i}(x)) \cap L_<(R_{W^j}, R_{W^j}(x)) \subseteq \mathrm{ri}(L_\leqslant(R_{W^j}, R_{W^j}(x))) \cap \mathrm{ri}(L_\leqslant(R_{W^j}, R_{W^j}(x)))$ we have that (see Remark 5.3.2 in Hiriart-Urruty and Lemaréhal (1993))

$$T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x) \cap T_{L_\leqslant(R_{W^j},R_{W^j}(x))}(x) = T_{I_{ij}^{\leqslant}(x)}(x)$$

and the result follows. □

Lemma 5.2
If we have that

$$\bigcap_{i=1}^{3} L_<(R_{W^i}, R_{W^i}(x)) \neq \emptyset \tag{16}$$

then

$$\mathrm{ri}\left(\bigcap_{i=1}^{3} T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x)\right) \neq \emptyset$$

$$\{0\} \notin \mathrm{ri}\left(\bigcap_{i=1}^{3} T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x)\right)$$

Proof
First, since $\bigcap_{i=1}^{3} T_{L_\leqslant(R_{W^i},R_{W^i}(x))}(x)$ is a pointed cone at 0 then its relative interior does not contain 0. By

(16) we have that $\bigcap_{i=1}^{3} \text{ri}(L_{\leqslant}(R_{W^i}, R_{W^i}(x))) \neq \emptyset$ then $\bigcap_{i=1}^{3} T_{L_{\leqslant}(R_{W^i},R_{W^i}(x))}(x) = T_{\bigcap_{i=1}^{3} L_{\leqslant}(R_{W^i},R_{W^i}(x))}(x)$ (see Hiriart-Urruty and Lemaréchal, 1993). On the other hand, since $\bigcap_{i=1}^{3} L_{\leqslant}(R_{W^i}, R_{W^i}(x)) \subseteq x + T_{\bigcap_{i=1}^{3} L_{\leqslant}(R_{W^i},R_{W^i}(x))}(x)$ then

$$\emptyset \neq \bigcap_{i=1}^{3} L_{<}(R_{W^i}, R_{W^i}(x)) \subseteq \text{ri}\left(\bigcap_{i=1}^{3} L_{\leqslant}(R_{W^i}, R_{W^i}(x))\right)$$
$$\subseteq \text{ri}\left(x + T_{\bigcap_{i=1}^{3} L_{\leqslant}(R_{W^i},R_{W^i}(x))}(x)\right)$$
$$= x + \text{ri}\left(T_{\bigcap_{i=1}^{3} L_{\leqslant}(R_{W^i},R_{W^i}(x))}(x)\right)$$

Thus, we conclude that $\text{ri}(T_{\bigcap_{i=1}^{3} L_{\leqslant}(R_{W^i},R_{W^i}(x))}(x)) \neq \emptyset$ and the result follows. □

Lemma 5.3
If we have that $I_{12}^{<}(x) \neq \emptyset$ then

$$I_{12}^{<}(x) \cap \mathscr{X}_{w-\text{par}}^{*}(W^1, W^2) \neq \emptyset$$

Proof
The set $I_{12}^{<}(x)$ is the set of points strictly dominating x. That means that any $y \in I_{12}^{<}(x)$ verifies $R_{W^i}(y) < R_{W^i}(x)$, $i = 1, 2$. Therefore, $\mathscr{X}_{w-\text{par}}^{*}(W^1, W^2) \cap I_{12}^{<}(x) \neq \emptyset$. □

The next result shows that the 3-criterion solution is nothing else but a kind of hull defined by the involved bicriteria solutions.

Definition 5.1
(See Figure 4). The curve $z(t), t \in [0, \infty)$ with $z(0) = x$ and $\lim_{t\to\infty} \|z(t)\| = +\infty$ separates the sets A and B, with respect to a convex cone Γ pointed at x, if

(a) $A, B \subset \Gamma$.
(b) There exists no continuous curve $y(t) \subset \Gamma, \forall t \in [0, 1]$ with $y(0) \in A$, $y(1) \in B$ and verifying that $\{z(t) : t \in (0, +\infty)\} \cap \{y(t) : t \in [0, 1]\} = \emptyset$.

Let us denote $\mathscr{X}_{w-\text{par}}^{*}(2) := \bigcup_{\substack{i,j \in \{1,2,3\} \\ i \neq j}} \mathscr{X}_{w-\text{par}}^{*}(W^i, W^j)$, the union of all bicriteria chains for three considered criteria.

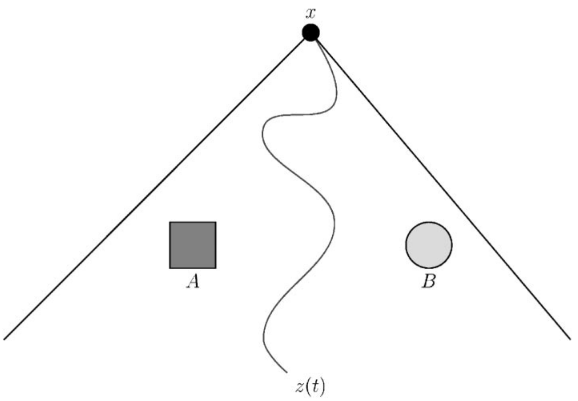

Figure 4. $z(t)$ separates the sets A and B with respect to the pointed cone at x.

Proposition 5.1

$$\mathscr{X}_{w-\text{par}}^{*}(W^1, W^2, W^3) = \text{encl}\big(\mathscr{X}_{w-\text{par}}^{*}(2)\big)$$

where $\text{encl}\big(\mathscr{X}_{w-\text{par}}^{*}(2)\big)$ is the bounded region whose boundary is $\mathscr{X}_{w-\text{par}}^{*}(2)$.

Remark 5.1
It is worth noting that the region $\text{encl}(\mathscr{X}_{w-\text{par}}^{*}(2))$ is well-defined because the set $\mathscr{X}_{w-\text{par}}^{*}(2)$ is connected (see Warburton, 1983). In addition, this region can be equivalently defined, as the set of points such that if $x \in \text{encl}(\mathscr{X}_{w-\text{par}}^{*}(2)) \backslash \mathscr{X}_{w-\text{par}}^{*}(2)$ there is no continuous curve $z(t)$, $t \in [0, \infty)$ with $z(0) = x$ and $\lim_{t\to\infty} \|z(t)\| = +\infty$, verifying that $z(t) \notin \mathscr{X}_{w-\text{par}}^{*}(2)$, $\forall t \in [0, \infty)$.

Proof
In order to prove that $\text{encl}(\mathscr{X}_{w-\text{par}}^{*}(2)) \subseteq \mathscr{X}_{w-\text{par}}^{*}(W^1, W^2, W^3)$, we note that $\mathscr{X}_{w-\text{par}}^{*}(W^i, W^j) \subseteq \mathscr{X}_{w-\text{par}}^{*}(W^1, W^2, W^3)\ \forall i, j \in \{1, 2, 3\}$. Now assume that there is a point x belonging to $\text{encl}(\mathscr{X}_{w-\text{par}}^{*}(2)) \backslash \mathscr{X}_{w-\text{par}}^{*}(2)$ and assume that x does not belong to $\mathscr{X}_{w-\text{par}}^{*}(W^1, W^2, W^3)$.

Since $x \notin \mathscr{X}_{w-\text{par}}^{*}(W^1, W^2, W^3)$ we have that $\bigcap_{i=1}^{3} L_{<}(R_{W^i}, R_{W^i}(x))(x) \neq \emptyset$. Then, by Lemma 5.2, $x - \text{ri}(\bigcap_{i=1}^{3} T_{L_{\leqslant}(R_{W^i},R_{W^i}(x))}(x)) \neq \{x\}$. Now, since $x \in \text{encl}\big(\mathscr{X}_{w-\text{par}}^{*}(2)\big) \backslash \mathscr{X}_{w-\text{par}}^{*}(2)$ and $x - \text{ri}(\bigcap_{i=1}^{3} T_{L_{\leqslant}(R_{W^i},R_{W^i}(x))}(x))$ is a cone pointed at x

then

$$S := x - \mathrm{ri}\left(\bigcap_{i=1}^{3} T_{L_{\leqslant}(R_{W^i}, R_{W^i}(x))}(x)\right) \cap \mathscr{X}^*_{w-\mathrm{par}}(2) \neq \emptyset$$

Let $y \in S$. Since $y \in x - \mathrm{ri}(\bigcap_{i=1}^{3} T_{L_{\leqslant}(R_{W^i}, R_{W^i}(x))}(x)) \subseteq \mathbb{R}^2 \backslash (\bigcup_{i=1}^{3} T_{L_{\leqslant}(R_{W^i}, R_{W^i}(x))}(x))$ then $R_{W^i}(x) < R_{W^i}(y)$, $i = 1, 2, 3$. Therefore, $y \notin \mathscr{X}^*_{w-\mathrm{par}}(W^1, W^2, W^3) \supseteq \mathscr{X}^*_{w-\mathrm{par}}(2)$ which contradicts that $y \in \mathscr{X}^*_{w-\mathrm{par}}(2)$.

Hence, we have that

$$\mathrm{encl}(\mathscr{X}^*_{w-\mathrm{par}}(2)) \subseteq \mathscr{X}^*_{w-\mathrm{par}}(W^1, W^2, W^3)$$

Now, let $x \in \mathscr{X}^*_{w-\mathrm{par}}(W^1, W^2, W^3)$. We must prove that $x \in \mathrm{encl}(\mathscr{X}^*_{w-\mathrm{par}}(2))$.

First, if there exists $i, j \in \{1, 2, 3\}$ such that $I_{ij}^{<}(x) = \emptyset$ then $x \in \mathscr{X}^*_{w-\mathrm{par}}(W^i, W^j) \subseteq \mathscr{X}^*_{w-\mathrm{par}}(2)$.

Second, if we have $I_{ij}^{<}(x) \neq \emptyset$, $\forall i, j \in \{1, 2, 3\}$, since $x \in \mathscr{X}^*_{w-\mathrm{par}}(W^1, W^2, W^3)$ then $\bigcap_{i=1}^{3} L_{<}(R_{W^i}, R_{W^i}(x)) = \emptyset$. Therefore, the conditions of Lemmas 5.1 and 5.3 are fulfilled (see Figure 5). This implies that

$$C_{ij} := I_{ij}^{<}(x) \cap \mathscr{X}^*_{w-\mathrm{par}}(W^i, W^j) \neq \emptyset \tag{17}$$

We must prove that there exists a chain of efficient points for two criteria surrounding the point x. We prove that by contradiction.

Assume that there exists a continuous curve $z(t)$, $t \in [0, \infty)$ such that (see Figure 6),

(a) $z(t)$ separates the sets C_{12} and C_{13} with respect to the cone $x + T_{L_{\leqslant}(R_{W^1}, R_{W^1}(x))}(x)$.
(b) $\mathscr{X}^*_{w-\mathrm{par}}(2) \cap (x + T_{L_{\leqslant}(R_{W^1}, R_{W^1}(x))}(x)) \cap \{z(t) : t \in [0, \infty)\} = \emptyset$.

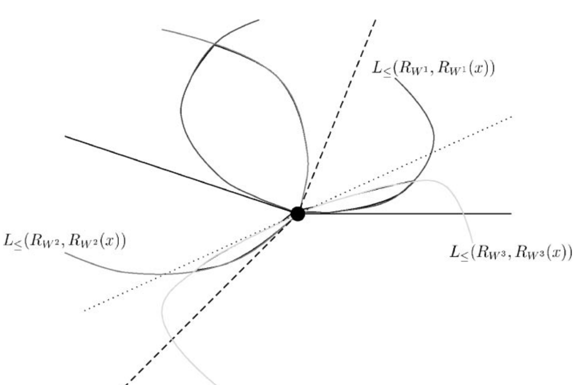

Figure 5. Case $x \in \mathscr{X}^*_{w-\mathrm{par}}(W^1, W^2, W^3) \backslash \mathscr{X}^*_{w-\mathrm{par}}(2)$.

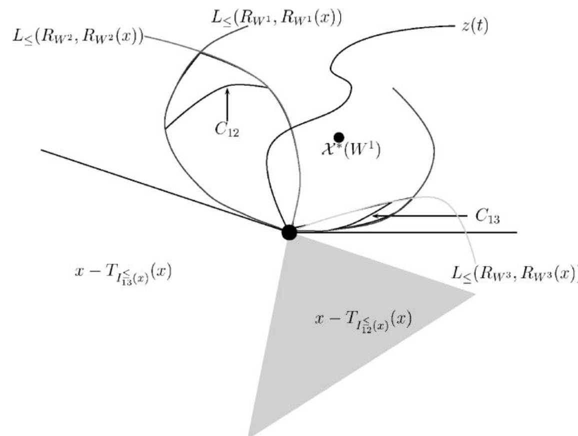

Figure 6. $z(t)$ separates the sets C_{12} and C_{13} with respect to the cone $x + T_{L_{\leqslant}(R_{W^1}, R_{W^1}(x))}(x)$.

First of all, $\mathscr{X}^*(W^1) \subseteq L_{\leqslant}(R_{W^1}, R_{W^1}(x)) \subseteq x + T_{L_{\leqslant}(R_{W^1}, R_{W^1}(x))}(x)$. In addition, we have that,

1.
$$\mathscr{X}^*(W^1) \cup C_{12} \subseteq \mathscr{X}^*_{w-\mathrm{par}}(W^1, W^2) \subseteq \mathbb{R}^2 \backslash (x - (T_{L_{\leqslant}(R_{W^1}, R_{W^1}(x))}(x) \cap T_{L_{\leqslant}(R_{W^2}, R_{W^2}(x))}(x)))$$

(by Lemma 5.1), and by Remark 5.3.2 (Hiriart-Urruty and Limaréchal, 1993) we also have that

$$T_{L_{\leqslant}(R_{W^1}, R_{W^1}(x))}(x) \cap T_{L_{\leqslant}(R_{W^2}, R_{W^2}(x))}(x) = T_{I_{12}^{\leqslant}(x)}(x).$$

2. $\mathscr{X}^*(W^1) \cup C_{13} \subseteq \mathscr{X}^*_{w-\mathrm{par}}(W^1, W^3) \subseteq \mathbb{R}^2 \backslash \left(x - T_{I_{13}^{\leqslant}(x)}(x)\right)$ (by Lemma 5.1).

This means that $\mathscr{X}^*_{w-\mathrm{par}}(W^1, W^2)$ and $\mathscr{X}^*_{w-\mathrm{par}}(W^1, W^3)$ cannot cross $x - T_{I_{12}^{\leqslant}(x)}(x)$ and $x - T_{I_{13}^{\leqslant}(x)}(x)$, respectively. On the other hand, we know that both $\mathscr{X}^*_{w-\mathrm{par}}(W^1, W^2)$ and $\mathscr{X}^*_{w-\mathrm{par}}(W^1, W^3)$ are connected sets containing $\mathscr{X}^*(W^1)$. Then, three cases can occur:

1. $\mathscr{X}^*(W^1)$ is separated from C_{12} by $z(t)$ then $\mathscr{X}^*_{w-\mathrm{par}}(W^1, W^2) \cap \{z(t) : t \in [0, \infty)\} \neq \emptyset$.
2. $\mathscr{X}^*(W^1)$ is separated from C_{13} by $z(t)$ then $\mathscr{X}^*_{w-\mathrm{par}}(W^1, W^3) \cap \{z(t) : t \in [0, \infty)\} \neq \emptyset$.
3. $\mathscr{X}^*(W^1) \cap \{z(t) : t \in [0, \infty)\} \neq \emptyset$.

Therefore, all of these three cases contradict the initial hypothesis.

We can use the same arguments with C_{12} and C_{23} as well as with C_{13} and C_{23} to obtain that the point x belongs to the region surrounded by the set of weakly efficient points for each of the two functions. □

As a direct consequence of the results of this section we get the following algorithm.

Algorithm 5.1

Input:
1. *Demand points* $A \subset \mathbb{R}^2$.
2. *Weight sets* $W^i = (w_a^i)_{a \in A},\ i = 1, \ldots, Q$.

Output:
1. $\mathscr{X}^*_{w-\text{par}}(W^1, \ldots, W^Q)$.

Steps
1. *Computation of the set* $\mathscr{X}^*_{\text{par}}(W^i, W^j)\ \forall i<j \in 1,2,3$
2. *Compute for all* $i,j,k \in \{1,\ldots,Q\} \mathscr{X}^*_{w-\text{par}}(W^i, W^j, W^k)$ *using Proposition* 5.1.
3. *Compute* $\mathscr{X}^*_{\text{par}}(W^1, \ldots, W^Q) = \bigcup_{i,j,k} \mathscr{X}^*_{w-\text{par}}(W^i, W^j, W^k)$.
4. *END* □

Example 5.1
We use the data of Example 4.1 and add two additional existing facilities $a_{17} = (14, 12)$ and $a_{18} = (15, 13)$.

Now we have three decision makers, each of them having two sets of weights:

$W^1 = \{w^{11}, w^{12}\}$ with

- $w^{11} = (1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$ and
- $w^{12} = (0,0,0,0,1,1,1,1,0,0,0,0,0,0,0,0,0,0)$;

$W^2 = \{w^{21}, w^{22}\}$ with

- $w^{21} = (0,0,0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,0)$ and
- $w^{22} = (0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,0,0)$.

$W^3 = \{w^{31}, w^{32}\}$ with

- $w^{31} = (1,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$ and
- $w^{32} = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1)$.

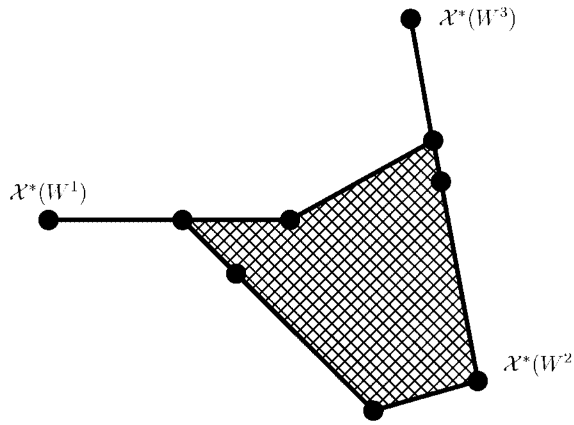

Figure 7. Illustration of Example 5.1. The bold part constitutes the set of Pareto solutions for all three criteria.

According to the results of this section, we compute the Pareto chain for all three bicriteria subproblems $\mathscr{X}^*_{\text{par}}(W^1, W^2)$, $\mathscr{X}^*_{\text{par}}(W^1, W^3)$, $\mathscr{X}^*_{\text{par}}(W^2, W^3)$. The result is shown in Figure 7. Note that according to the results obtained also the marked enclosed region is Pareto optimal.

The algorithms in this paper are implemented with LOLA (Hamacher *et al.*, 2000) and the program code is available upon request from lola@itwm.fhg.de.

6. CONCLUSIONS

In this paper, we have shown how to derive an efficient algorithm for a robustness concept in multi-criteria location. An emphasis was put on the geometrical structure of this multi-criteria model. Extensions to higher dimensions, to other distance measures and to more general objective functions seem to be natural and are currently under research.

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